Nonminimal coupling, equivalence principle and exact Foldy-Wouthuysen transformation

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Abstract

It is shown that the exact Foldy-Wouthuysen transformation for spin-0 particles on spacetimes described by the metrics $ds^2 = V^2 dt^2 - W^2 d\mathbf{x}^2$, where $V = V(\mathbf{x})$ and $W = W(\mathbf{x})$, only exists if the scalar field is nonminimally coupled to the Ricci scalar field with a coupling constant equal to $\frac{1}{6}$. The nonminimal coupling term, in turn, does not violate the equivalence principle. As an application we obtain the nonrelativistic Foldy-Wouthuysen Hamiltonian concerning the general solution to the linearized field equations of higher-derivative gravity for a static pointlike source in the Teyssandier gauge.

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1 Introduction

The COW experiment [1] as well as the Bonse-Wroblewski one [2] not only shed a new light on the physical phenomena in which both gravitational and quantum effects are interwoven but also showed that the aforementioned phenomena are nomore beyond our reach. The theoretical analysis concerning these experiments consisted simply in inserting the Newtonian gravitational potential into the Schrödinger equation. To improve this analysis we need to learn, certainly, how to obtain an adequate interpretation for relativistic wave equations in curved space. In other words, we have to acquaint ourselves with the issue of the gravitational effects on quantum mechanical systems. This can be done by constructing the Foldy-Wouthuysen transformation (FWT)[3, 4] - the keystone of relativistic quantum mechanics - for both bosons and fermions coupled to the spacetime metric. Here we address ourselves to the problem of finding the exact FWT for spin-0 particles coupled to the static metrics

$$ds^2 = V^2 dt^2 - W^2 d\mathbf{x}^2 \quad , \tag{1.1}$$

where $V = V(\mathbf{x})$ and $W = W(\mathbf{x})$.

We employ natural units in which $c = \hbar = 1$. Our convention are to use a metric with signature $(+---)$ and define the Riemann and Ricci tensors as $R^\rho{}_{\lambda\mu\nu} = -\partial_\nu \Gamma^\rho{}_{\lambda\mu} + \partial_\mu \Gamma^\rho{}_{\lambda\nu}$ $-\Gamma^\sigma{}_{\lambda\mu}\Gamma^\rho{}_{\sigma\nu} + \Gamma^\sigma{}_{\lambda\nu}\Gamma^\rho{}_{\sigma\mu}$ and $R_{\mu\nu} = R^\rho{}_{\mu\nu\rho}$.

2 Exact FWT for spin 0 particle

The propagation of a free scalar ϕ with mass m in Minkowski spacetime is described by the Klein-Gordon (KG)

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\phi = 0 \quad , \tag{2.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric.

A generalization of (2.1) to a curved background is

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2 + \lambda R\right)\phi = 0 \quad , \tag{2.2}$$

where the numerical constant λ represents the direct coupling between the field and the curvature. Since the coupling constant λ can have any real value, we are on the horns of a dilemma: Do we pick out the value $\lambda = 0$, which means that the field and the curvature are minimally coupled, or do we choose $\lambda \neq 0$, implying in a nonminimal coupling of the field to the curvature? Fortunately, there are two strong arguments that seems to favour the choice $\lambda = \frac{1}{6}$:

1. The equation for the massless scalar field is conformally invariant[5]-[7].

2. Under the assumptions that *(i)* the scalar field satisfies (2.2), and *(ii)* the field ϕ does not violate the equivalence principle, λ is forced to assume the value $\frac{1}{6}$ [8]-[10].

We assume thus that ϕ satisfies the covariant KG equation

$$\left(\Box + m^2 + \frac{1}{6}R\right)\phi = 0 \quad ,$$

where

$$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right) \quad .$$

Specializing now to the static metrics (1.1) we find

$$\ddot{\phi} - F^2\nabla^2\phi - F^2\nabla\ln(VW)\cdot\nabla\phi + m^2V^2\phi + \frac{1}{6}R\phi^2 = 0 \quad ,$$

where

$$R = \frac{2}{W^4}(\nabla W)^2 - \frac{2}{VW^3}\nabla V\cdot\nabla W - \frac{2}{VW^2}\nabla^2 V - \frac{4}{W^3}\nabla^2 W \quad .$$

Here $F^2 \equiv \frac{V^2}{W^2}$ and the differentiation with respect to time is denoted by dots.

In order to understand the physics of the equation above we write it in first order form

$$i\dot{\Phi} = \mathcal{H}\Phi \quad ,$$

where

$$\begin{aligned}\Phi &= \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \quad ,\\ \phi_1 &= \frac{1}{2}\left(\phi + \frac{i}{m}\dot{\phi}\right) \ , \ \phi_2 = \frac{1}{2}\left(\phi - \frac{i}{m}\dot{\phi}\right) \quad ,\end{aligned}$$

with the Hamiltonian given by

$$\mathcal{H} = \frac{m}{2}\xi^T - \xi\theta \quad , \tag{2.3}$$

where

$$\begin{aligned}\xi &= \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix} \quad ,\\ \theta &= \frac{F^2}{2m}\nabla^2 - \frac{F^2}{2m}\nabla\ln(VW)\cdot\nabla - \frac{m}{2}V^2 - \frac{1}{24m}V^2R \quad .\end{aligned}$$

The matrix ξ has the following algebraic properties

$$\xi^2 = 0 \quad , \quad \left\{\xi, \xi^T\right\} = 4 \quad .$$

Redefining the scalar field and the Hamiltonian

$$\Phi' = f\Phi \quad , \quad \mathcal{H}' = f\mathcal{H}f^{-1} \quad ,$$

with

$$f = V^{-1/2}W^{3/2} \quad ,$$

we obtain a new Hamiltonian which is explicitly Hermitian with respect to the usual flat space measure:

$$\mathcal{H}' = \frac{m}{2}\xi^T - \xi\theta' \quad ,$$

where

$$\begin{aligned} \theta' &= f\theta f^{-1} \\ &= -\frac{m}{2}V^2 - \frac{F}{2m}\hat{\mathbf{p}}^2 F + \frac{1}{8m}\nabla F\cdot\nabla F - \frac{1}{12m}F\nabla^2 F \quad . \end{aligned}$$

Here $\hat{\mathbf{p}} = -i\nabla$ denotes the momentum operator.

It is astonishing and at the same time fascinating that the square of the transformed Hamiltonian $\mathcal{H}'$,

$$\mathcal{H}^{\Box} = -\frac{m}{2}\theta'\left\{\xi, \xi^T\right\} = -2m\theta' I \quad ,$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ,$$

is diagonal and, obviously, an even operator. Formally, we have

$$\sqrt{\mathcal{H}^{\Box}} = (-2m\theta')^{1/2}\eta \quad ,$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \eta^2 = 1 \quad .$$

Accordingly, $\mathcal{H}' = f\mathcal{H}f^{-1}$ is the exact FWT for the KG equation in curved space [11].

Now, taking into account, that the Hamiltonian squared can be rewritten as

$$\mathcal{H}^{\Box} = m^2V^2 + F\hat{p}^2F - \frac{1}{4}\nabla F\cdot\nabla F + \frac{1}{6}F\nabla^2 F \quad ,$$

we promptly obtain the quasirelativistic Hamiltonian simply by assuming that the dominating term is m:

$$\sqrt{\mathcal{H}^{\Box}} \approx mV + \frac{1}{4m}\left(W^{-1}\hat{p}^2F + F\hat{p}^2W^{-1}\right) - \frac{1}{8mV}\nabla F\cdot\nabla F + \frac{1}{12mW}\nabla^2 F \quad . \tag{2.4}$$

Two comments are in order here:

(i) The preceding Hamiltonian contains the gravitational Darwin term

$$\frac{1}{12mW}\nabla^2 F \quad ,$$

which is lacking in all the works on this subject [12] with the exception of that by Obukhov [13].

(ii) (2.4) is identical to the spinless sector found in Ref. [13] for the Dirac particle except for the Darwin term which is one third of the corresponding term in the fermionic case. Fermions and bosons, of course, lead to different Darwin terms [14].

3 The nonrelativistic FW Hamiltonian concerning the general solution to the linearized field equations of higher-derivative gravity for a static pointilike source in the Teyssandier gauge

Higher-derivative gravity is defined by the action

$$S = \int d^4x\sqrt{-g}\left[\frac{2R}{\kappa^2} + \frac{\alpha}{2}R^2 + \frac{\beta}{2}R^2_{\mu\nu}\right] - \int d^4x\sqrt{-g}\mathcal{L}_M \quad ,$$

where $\mathcal{L}_M$ is the Lagrangian density for matter, $\kappa^2 \equiv 32\pi G$, with G being the Newton's constant, and α and β are dimensionless parameters. The corresponding field equations are

$$\begin{aligned}\frac{2}{\kappa^2}G_{\mu\nu} &+ \frac{\alpha}{2}\left[-\frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu} + 2\nabla_\mu\nabla_\nu R - 2g_{\mu\nu}\Box R\right] \\ &+ \frac{\beta}{2}\left[-\frac{1}{2}g_{\mu\nu}R^2_{\rho\sigma} + \nabla_\mu\nabla_\nu R + 2R_{\mu\rho\lambda\nu}R^{\rho\lambda} - \frac{1}{2}g_{\mu\nu}\Box R - \Box R_{\mu\nu}\right] + \frac{1}{2}T_{\mu\nu} = 0 \quad (3.1)\end{aligned}$$

where

$$\delta\int d^4x\sqrt{-g}\mathcal{L}_M \equiv \int \sqrt{-g}d^4x\frac{T^{\mu\nu}}{2}\delta g_{\mu\nu} \quad .$$

In the weak field approximation, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \quad ,$$

and in the Teyssandier gauge [15], (3.1) reduces to

$$\left(1 - \frac{\beta\kappa^2}{4}\Box\right)\left(-\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{6}\eta_{\mu\nu}\bar{R}\right) = \frac{\kappa}{4}\left[T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu}\right] \quad ,$$

where

$$\bar{R} = \frac{1}{2}\Box h - \gamma^{\mu\nu}{}_{,\mu\nu} \quad ,$$

with $\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. Here indices are raised (lowered) using $\eta^{\mu\nu}(\eta_{\mu\nu})$. Teyssandier showed that the general solution of the above equations is given by [15]

$$h_{\mu\nu} = h^{(E)}_{\mu\nu} + \psi_{\mu\nu} - \Phi\eta_{\mu\nu} \quad ,$$

where $h^{(E)}_{\mu\nu}$, $\psi_{\mu\nu}$and Φ satisfy the following equations:

$$\begin{aligned}\Box h^{(E)}_{\mu\nu} &= \frac{\kappa}{2}\left[\frac{T\eta_{\mu\nu}}{2} - T_{\mu\nu}\right] \ , \ \gamma^{(E),\nu}_{\mu\nu} = 0 \ , \ \gamma^{(E)}_{\mu\nu} = h^{(E)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{(E)}; \\ \left(\Box + m_1^2\right)\psi_{\mu\nu} &= \frac{\kappa}{2}\left[T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu}\right] \ , \ \psi^{\mu\nu}{}_{,\mu\nu} - \Box\psi = 0 \ ; \ \left(\Box + m_0^2\right)\Phi = \frac{\kappa T}{12} \quad .\end{aligned}$$

Here $m_0^2 \equiv \frac{2}{\kappa^2(3\alpha+\beta)}$, $m_1^2 \equiv -\frac{4}{\kappa^2\beta}$.

For a point particle of mass M located at $\mathbf{r} = \mathbf{0}$, the general solution of the equations above is [16, 15, 17]

$$g_{00} = 1 + 2MG\left[-\frac{1}{r} - \frac{1}{3}\frac{e^{-m_0 r}}{r} + \frac{4}{3}\frac{e^{-m_1 r}}{r}\right] \quad , \tag{3.2}$$

$$g_{11} = g_{22} = g_{33} = -1 + 2MG\left[-\frac{1}{r} + \frac{2}{3}\frac{e^{-m_1 r}}{r} + \frac{1}{3}\frac{e^{-m_0 r}}{r}\right] \quad . \tag{3.3}$$

It is worth mentioning that we have assumed that m_0^2 and m_1^2 are real, in order to assure asymptotic agreement of the theory with Newton's law.

From (3.2) and (3.3) we get immediately

$$V = 1 + MG\left[-\frac{1}{r} - \frac{1}{3}\frac{e^{-m_0 r}}{r} + \frac{4}{3}\frac{e^{-m_1 r}}{r}\right] \quad , \tag{3.4}$$

$$W = 1 - MG\left[-\frac{1}{r} + \frac{2}{3}\frac{e^{-m_1 r}}{r} + \frac{1}{3}\frac{e^{-m_0 r}}{r}\right] \quad . \tag{3.5}$$

Inserting (3.4) and (3.5) into (2.4) we come to the conclusion that the nonrelativistic FW Hamiltonian is given by

$$\begin{aligned} \sqrt{\mathcal{H}^{\Box}} &= \Bigg\{ m + \left(1 + \frac{e^{-m_0 r}}{3} - \frac{4}{3}e^{-m_1 r}\right) m\, \mathbf{g}\cdot\mathbf{x} + \frac{\hat{\mathbf{p}}^2}{2m} + \frac{1}{2m}\hat{\mathbf{p}}.u\hat{\mathbf{p}} \\ &+ \frac{1}{2m}\nabla\cdot\Bigg[\left(\frac{7}{6} - (1+m_1 r)\, e^{-m_1 r} - \frac{1}{6}(1+m_0 r)\, e^{-m_0 r}\right)\mathbf{g}\Bigg]\Bigg\}\eta \quad , \end{aligned} \tag{3.6}$$

with

$$\mathbf{g} = -GM\frac{\mathbf{r}}{r^3}$$

and

$$u \equiv MG\left(-\frac{3}{r} + \frac{1}{3}\frac{e^{-m_0 r}}{r} + \frac{8}{3}\frac{e^{-m_1 r}}{r}\right)$$

.

Note that (3.6) tends to

$$\sqrt{\mathcal{H}^{\Box}} = \left[m + m\,\mathbf{g}\cdot\mathbf{x} + \frac{\hat{\mathbf{p}}^2}{2m} + \frac{3}{2m}\hat{\mathbf{p}}\cdot(\mathbf{g}\cdot\mathbf{x})\hat{\mathbf{p}} + \frac{7}{12m}\nabla\cdot\mathbf{g}\right]\eta \quad , \tag{3.7}$$

as $m_0, m_1 \to \infty$, which is nothing but the nonrelativistic Hamiltonian for the spin-0 particle in the external gravitational field of a central gravitating body of mass M in the framework of ordinary general relativity. The first terms of this expansion has been found in [18] by means of another method. The coefficient in the last term of (3.7) differs from that of [13] since the Darwin term contribution in the scalar case enters as 1/3 of the relevant term for fermions. It is worth mentioning that we have neglected in (3.6) and (3.7) the higher order relativistic and gravitational/inertial terms.

4 The nonminimal coupling term does not violate the equivalence principle

Let us now show that the nonminimal coupling term does not violate the equivalence principle for spin-0 particles by making a comparison of the true gravitational coupling with the pure inertial case. To do that, we recall that in the case of the flat Minkowski space in accelerated frame,

$$V = 1 + \mathbf{a}\cdot\mathbf{x} \quad , \quad W = 1 \quad .$$

Consequently, the corresponding nonrelativistic FW Hamiltonian is

$$\sqrt{\mathcal{H}^{\Box}} = \left[m + m\,\mathbf{a}\cdot\mathbf{x} + \frac{\hat{\mathbf{p}}^2}{2m} + \frac{1}{2m}\hat{\mathbf{p}}\cdot(\mathbf{a}\cdot\mathbf{x})\hat{\mathbf{p}} \right]\eta \quad , \tag{4.1}$$

where we have also neglected the higher order relativistic and gravitational/inertial terms.

For the particle m far away from the body M one can neglect the terms $\frac{3}{2m}\hat{\mathbf{p}}\cdot(\mathbf{g}\cdot\mathbf{x})\hat{\mathbf{p}}$ and $\frac{1}{2m}\hat{\mathbf{p}}\cdot(\mathbf{a}\cdot\mathbf{x})\hat{\mathbf{p}}$ in (3.7) and (4.1), respectively, since they are of the order $(GM)/(mr)$. In (4.1) we are assuming that $\mathbf{a}$ is such that $|\mathbf{a}\cdot\mathbf{x}|/m$ is of the order $(GM)/(mr)$. Then, we come to the conclusion that the covariant KG theory with $\lambda = \frac{1}{6}$ is in agreement with the equivalence principle.

5 Final remarks

We have shown that *(i)* the KG equation only admits exact FWT if $\lambda = \frac{1}{6}$, and *(ii)* the nonminimal coupling term, contrary to a previous claim [19], does not violate the equivalence principle. We are not claiming, however, that $\lambda = \frac{1}{6}$ is the correct coupling for the various scalar particles. The question of which value(s) of λ should constitute the correct coupling(s) to gravity depends on the particular quantum field theory used for the scalar field ϕ (see e.g. [20] and references therein).

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